Christ Church
Grammar School

2020
TEST 4

MATHEMATICS SPECIALIST Year 12

Section One:
Calculator-free

Your name SOLUTIONS

Teacher's name ______________________________

Time and marks available for this section

Reading time for this section: 2 minutes
Working time for this section: 30 minutes
Marks available: 26 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

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3. Answer all questions.

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6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

Question 1 **(6 marks)**

If $y^4 = x^4 - 4xy - 7$, then determine

(a) $\frac{dy}{dx}$. (3 marks)

$$4y^3 \frac{dy}{dx} = 4x^3 - \left(4y + 4x\frac{dy}{dx}\right) \checkmark$$

differentiates implicitly correctly

collects like terms

$$(4y^3 + 4x)\frac{dy}{dx} = 4x^3 - 4y \checkmark$$

$$\frac{dy}{dx} = \frac{4(x^3 - y)}{4(y^3 + x)} = \frac{x^3 - y}{y^3 + x}. \checkmark$$

expresses $\frac{dy}{dx}$ in simplified form

(b) the value of gradient at the point $(2,1)$. (1 mark)

$$\frac{dy}{dx} = \frac{2^3 - 1}{1^3 + 2}$$

$$= \frac{7}{3} \checkmark$$

obtains correct value

(c) the equation of the normal at the point $(2,1)$. (2 marks)

If $m = \frac{7}{3}$ at $(2,1)$

then $\perp = -\frac{3}{7}$ $\checkmark$

states $\perp$ gradient

$$y = -\frac{3}{7}x + c$$

$$1 = -\frac{3}{7}(2) + c$$

$$1 + \frac{6}{7} = c$$

$$\frac{13}{7} = c$$

$$\therefore\ y = -\frac{3}{7}x + \frac{13}{7} \checkmark$$

determines/obtains equation of normal

or $7y = -3x + 13$ or $7y + 3x = 13$

Question 2 **(8 marks)**

(a) Differentiate $y = (\sin x)^x$, using logarithmic differentiation. (3 marks)

$$\ln y = x\ln(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{\cos x}{\sin x} + 1.\ln(\sin x)$$ ✓ differentiates correctly

$$\frac{dy}{dx} = \left[x.\frac{\cos x}{\sin x} + \ln(\sin x)\right]\underbrace{(\sin x)^x}_{y}$$ ✓ ✓ obtains dy/dx in terms of 'x' & 'y'

$$\text{or } \frac{dy}{dx} = \left[x.\cot x + \ln(\sin x)\right].(\sin(x))^x$$ obtains $\frac{dy}{dx}$ in terms of x only

(b) If $\frac{dA}{dt} = 6e^{2t}$ and $A = 4$ when $t = 0$, then

(i) determine A in terms of t. (2 marks)

$$A = 3e^{2t} + c$$ ✓ obtains $A = \quad + c$

$$4 = 3 + c$$

$$c = 1$$

$$\therefore \quad A = 3e^{2t} + 1$$ ✓ obtains correct equation

(ii) determine the exact value of A when $t = 0.5$. (1 mark)

$$A = 3e^{2(0.5)} + 1$$

$$A = 3e + 1$$ ✓ obtains correct <u>exact</u> answer

(iii) use $\frac{dA}{dt}$ to determine the approximate change in A when t changes from 0 to 0.01. (2 marks)

$$\frac{dA}{dt} \approx \frac{\delta A}{\delta t}$$

$$\delta A \approx \frac{dA}{dt} \times \delta t$$

$$\approx 6e^{2t} \times \frac{1}{100}$$ ✓ uses incremental formula and substs. in $\frac{1}{100}$.

$$= 6e^0 \times \frac{1}{100} = \frac{6}{100} \text{ or } 0.06$$

$$= \frac{3}{50}.$$ ✓ obtains correct value.

Question 3 **(4 marks)**

Determine the exact values) of t, on the curve $x = t(t-4), y = t^2(t-4)$ at which the gradient is $\frac{1}{2}$.

$x = t^2 - 4t \quad y = t^3 - 4t^2$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dx}{dt} = 2t - 4$$

$$\frac{dy}{dt} = 3t^2 - 8t$$

$$\frac{dy}{dx} = (3t^2 - 8t) \times \frac{1}{2t-4} \quad ✓$$

correctly determines $\frac{dy}{dx}$

$$\frac{1}{2} = \frac{3t^2 - 8t}{2t - 4} \quad ✓$$

subst $y' = \frac{1}{2}$ for $\frac{dy}{dx}$

$$\frac{2(t-2)}{2} = 3t^2 - 8t$$

$$0 = 3t^2 - 9t + 2 \quad ✓$$

simplifies expression to $= 0$

$a = 3, \quad b = -9, \quad c = 2$

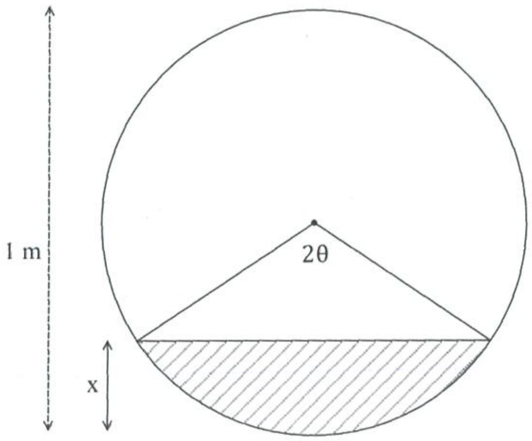
$$t = \frac{9 \pm \sqrt{81 - 4 \times 3 \times 2}}{6}$$

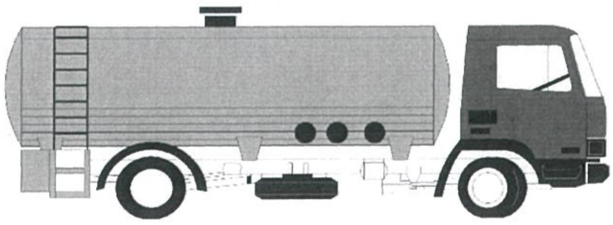
$$= \frac{9 \pm \sqrt{57}}{6}. \quad ✓$$

solves for 't' correctly.

Question 4 (10 marks)

A raw milk tank truck has a cylindrical shaped tank that is 1 m in diameter and 8 m in length. The tank is being filled from empty to half-full at a rate of 0.1 m^3 of raw milk per minute. The diagram below shows a cross section of the tank, where x is the depth of the raw milk level.





(a) Show that the volume V $[m^3]$ of raw milk in the tank is given by $V = 2\theta - \sin(2\theta)$, where $0 \le \theta \le \frac{\pi}{2}$. (2marks)

area of segment $= \frac{r^2}{2}(\theta - \sin\theta)$

$= \frac{\left(\frac{1}{2}\right)^2}{2}(2\theta - \sin 2\theta)$ ✓ obtains area of segment

$= \frac{1}{8}(2\theta - \sin 2\theta)$ ✓ shows calculation for V

$V = A \times h$

$= 8 \times \frac{1}{8}(2\theta - \sin 2\theta) = (2\theta - \sin 2\theta)$

(b) Using the equation in (a), determine an expression for $\frac{dV}{d\theta}$ in terms of θ. (1 mark)

$\frac{dV}{d\theta} = 2 - 2\cos 2\theta$ ✓ differentiates correctly

Question 4 continued

(c) The diagram below shows a way to link the rising depth of the raw milk x with the angle θ and the radius of the circle. Use this diagram to express x in terms of θ, and hence show that: (3 marks)

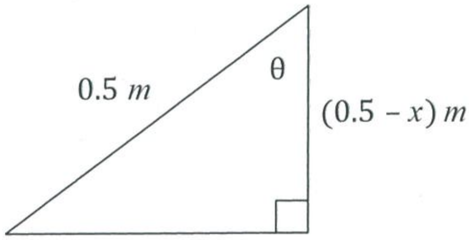
$$\frac{dx}{dt} = \frac{1}{2}\sin\theta\,\frac{d\theta}{dt}$$

$\cos\theta = \dfrac{0.5-x}{0.5}$

$= \dfrac{\frac{1}{2}-x}{\frac{1}{2}}$

$= \left(\frac{1}{2}-x\right)\times 2$

$= 1-2x$ ✓ determines $\cos\theta$ in terms of x



$\therefore\ x = \dfrac{1-\cos\theta}{2}$ ✓ determines $x =$

or $x = \frac{1}{2} - \frac{1}{2}\cos\theta$

$\dfrac{dx}{dt} = \dfrac{dx}{d\theta} \times \dfrac{d\theta}{dt}$

$= -\frac{1}{2} \times -\sin\theta \times \dfrac{d\theta}{dt}$ ✓ differentiates $\frac{dx}{d\theta}$ correctly + uses chain rule

$= \frac{1}{2}\sin\theta \cdot \dfrac{d\theta}{dt}$

(d) Determine the exact rate at which the raw milk level is rising at the instant when the raw milk is 25 cm deep. (4 marks)

$\dfrac{d\theta}{dt} = \dfrac{d\theta}{dV} \times \dfrac{dV}{dt}$

$= \dfrac{1}{2-2\cos 2\theta} \times 0.1$

when $\theta = \pi/3$ $\dfrac{d\theta}{dt} = \dfrac{0.1}{2-2\cos 2\pi/3}$ ✓ uses chain rule to determine $\frac{d\theta}{dt}$

$= \dfrac{0.1}{2+1}$

$= \dfrac{0.1}{3}$

$= \dfrac{1}{30}$ ✓ determines value of $\frac{d\theta}{dt}$ when $\theta = \pi/3$

when $x = 0.25$

$\cos\theta = 0.5$

$\theta = \pi/3$ ✓ determines θ, when $x = 0.25$

$\dfrac{dx}{dt} = \dfrac{1}{2}\sin\theta \cdot \dfrac{d\theta}{dt}$

$= \frac{1}{2}\sin(\pi/3) \times \frac{1}{30}$

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} \times \frac{1}{30}$

$= \dfrac{\sqrt{3}}{120}$ m³/min. ✓ calculates $\frac{dx}{dt}$ as exact value.

End of questions

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MATHEMATICS SPECIALIST Year 12

Section Two:
Calculator-assumed

Your name SOLUTIONS

Teacher's name ______________________

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Reading time for this section: 2 minutes
Working time for this section: 45 minutes
Marks available: 45 marks

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Question 5 **(6 marks)**

The "Red CAT" bus runs to an approximate schedule of 10 minutes between busses in the city centre. The time T in minutes that models the time interval between two buses (i.e. the time it takes for one bus to arrive after another one has departed) is modelled by a uniform distribution where $0 \leq T \leq 20$.
The population mean is $\mu(T) = 10$ minutes and the population variance is $\sigma^2(T) = 300$.
A sample of 30 bus intervals is taken and the sample mean $\bar{T}$ is calculated.

(a) Determine $P(5 \leq \bar{T} \leq 15)$ correct to **two (2)** decimal places. (3 marks)

$\bar{T} \sim N\left(10, \frac{300}{30}\right)$ or $\bar{T} \sim N(10, 10)$ ✓ makes statement re $\bar{T} \sim$ Normal

$\therefore \sigma(\bar{T}) = \sqrt{10}$ ✓ obtains value for st. dev

$P(5 \leq \bar{T} \leq 15) = 0.8862$

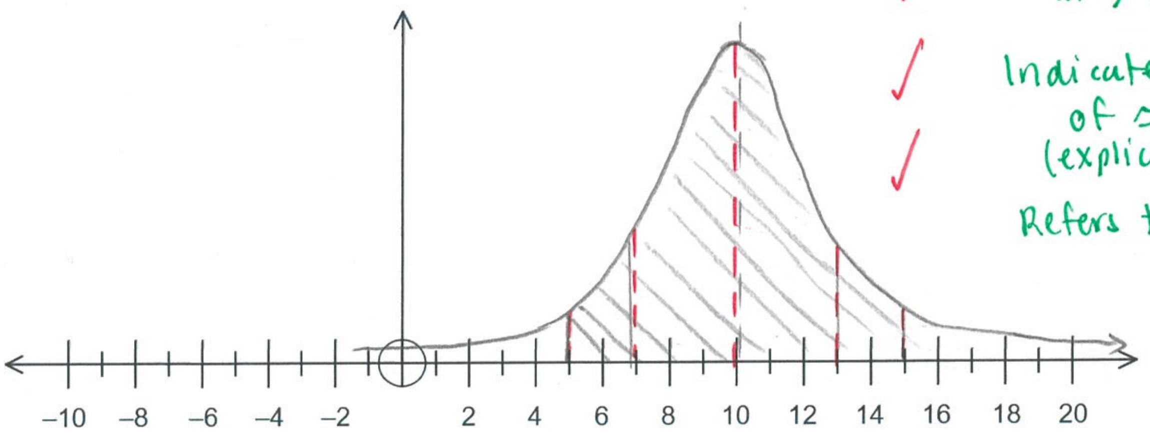
$= 0.89$ (2dp) ✓ calculates correct probability to 2 dp.

(b) A large number of samples are taken, each with 30 bus interval measurements. Sketch the likely distribution of the sample mean $\bar{T}$. Indicate in your diagram the calculation from part (a). (3 marks)

✓ Normal distribution at $\mu(T) = 10$

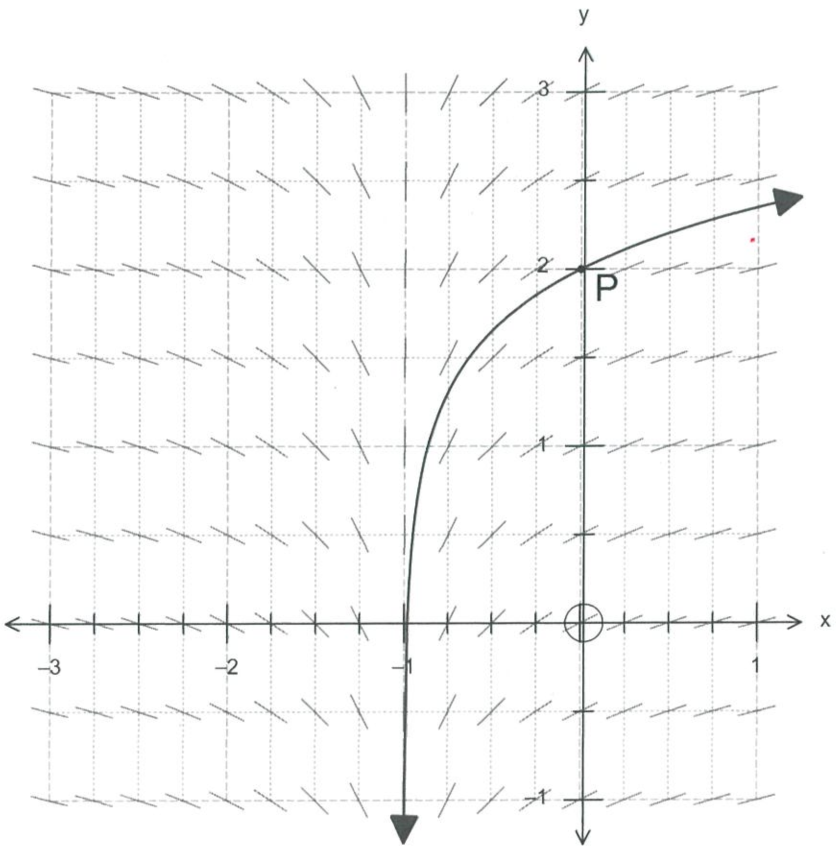
✓ Indicates a st. dev of $\simeq$ 3 min (explicit or implicit)

✓ Refers to shaded area from part (a).



Question 6 **(6 marks)**

The slope field shown below is given by $\frac{dy}{dx} = \frac{1}{2x+2}$ with $x \neq -1$.



(a) Determine the value of the slope field at the point $P(0, 2)$ shown. (1 mark)

When $x = 0$ $\frac{dy}{dx} = \frac{1}{(0)(2)+2}$

$= \frac{1}{2}$ ✓

obtains correct value

(b) Explain the behaviour of the slope field along the line $x + 1 = 0$. (1 mark)

$x + 1 = 0$

$\Rightarrow x = -1$

$\therefore$ slope is undefined ✓
as $x = -1$ is a vertical asymptote.

makes correct statement

Question 6 continued

(c) Determine the equation of the solution shown that passes through $P(0, 2)$.
(4 marks)

$$\frac{dy}{dx} = \frac{1}{2(x+1)}$$

$$\Rightarrow \int 2\,dy = \int \frac{dx}{x+1}$$ ✓

separates variables & places $\int$ correctly

$$2y = \ln|x+1| + c$$ ✓

Performs correct integration

$$y = \frac{1}{2}\ln|x+1| + c$$

when $x=0$, $y=2$

$$2 = \frac{1}{2}\ln|1| + c$$

$$2 = \frac{1}{2}\times 0 + c$$

$$\therefore c = 2$$ ✓

obtains value for 'c'

$$y = \frac{1}{2}\ln|x+1| + 2$$ ✓

obtains explicit expression for y.

or

$$y = \ln\sqrt{x+1} + 2.$$

Question 7 **(8 marks)**

(a) The displacement, x metres, of an object from an origin, O, is given by $x = A\cos(kt)$, where A and k are constants. Prove that the object is moving with simple harmonic motion and that it is initially at an extreme position. (3 marks)

$$\frac{dx}{dt} = -kA\sin(kt)$$

$$\frac{d^2x}{dt^2} = -k^2A\cos(kt) \checkmark$$

correctly differentiates function twice

$$= -k^2x \checkmark$$

correctly makes statement $-k^2x$.

$\Rightarrow$ SHM.

when $t=0$, $x=A$ ✓ $\therefore$ starts at the extreme.

shows starts at extreme.

Question 7 continued

(b) The depth of water in a harbour, above and below the mean depth, is an example of simple harmonic motion. In a harbour, the low tide depth of 3 metres is recorded at 7 a.m. one morning and the next high tide is expected to record a depth of 15 metres at 1:20 p.m. later that same day. A container ship requires a depth of at least 5 metres for safe entry into the harbour, for unloading at the dockside and for leaving. Determine, to the nearest 5 minutes, the times between 7 a.m. and 9 p.m. that day, for which it is safe for the ship to engage in these activities. (5 marks)

Low tide @ 7am → 3m

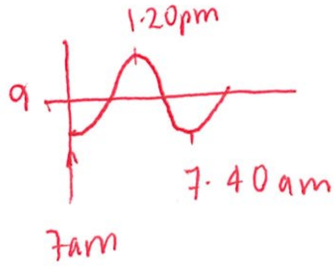
High tide @ 1:20pm → 15m

$\therefore A = \frac{15-3}{2} = 6\text{m}$ ✓ obtains amplitude + centre line.

mean position $= \frac{15+3}{2} = 9\text{m}.$

Assume $t=0$ at 7am.

$x = -6\cos(kt) + 9.$



repeats every $12\frac{2}{3}$ hrs

correctly calculates period ✓

$T = \frac{2\pi}{\frac{38}{3}} = \frac{3\pi}{19}.$

$\therefore x = -6\cos\left(\frac{3\pi}{19}t\right) + 9$ ✓ obtains correct expression/equation

When $x=5$

$5 = -6\cos\left(\frac{3\pi t}{19}\right) + 9$

$\therefore t = 1.7$ and 11 ✓ solves for t

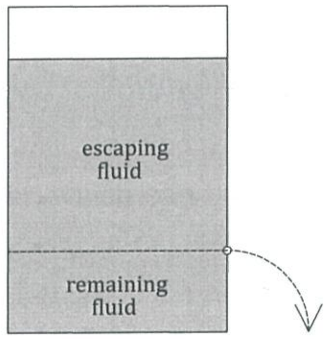
When $t = 1.7 \equiv$ 8.40am

When $t = 11 \equiv$ 6pm

$\therefore$ Safe passage from 8.40 am to 6pm. ✓ obtains correct time for safe passage to nearest 5mins

Question 8 **(7 marks)**

The diagram represents a container full of fluid that is punctured on the side. The fluid above the puncture line escapes due to pressure and gravity while the fluid below the line remains in the container.



The rate at which the fluid escapes is proportional to the amount of fluid left in the container, which can be modelled by the differential equation:

$$\frac{dQ}{dt} = k(100 - Q)$$

where Q represents the amount of fluid left in the container, in litres, and t is the time, in minutes, since the fluid began to leak.

(a) Use calculus to show that the solution to this differential equation is given by $Q(t) = Ae^{-kt} + 100$, where A is a constant. (3 marks)

$$\frac{dQ}{dt} = k(100-Q) = -k(Q-100)$$

$$\therefore \int \frac{dQ}{Q-100} = -\int k\,dt$$

$$\ln|Q-100| = -kt + c$$

$$Q - 100 = e^{-kt+c}$$

$$= e^{c} \cdot e^{-kt}$$

$$\therefore \quad Q = Ae^{-kt} + 100$$

correctly separates variables ✓

integrates correctly ✓

obtains expression for Q. ✓

or

$$\int \frac{dQ}{100-Q} = \int k\,dt$$

$$-\ln|100-Q| = kt + c$$

$$\ln|100-Q| = -kt + c$$

$$100 - Q = e^{-kt+c}$$

$$= e^{-kt} \cdot e^{c}$$

$$Q = 100 - e^{c} \cdot e^{-kt}$$

$$= 100 + e^{-c} \cdot e^{-kt}$$

$$\therefore \quad Q = 100 + A \cdot e^{-kt}$$

Question 8 continued

The container originally holds 900 litres of fluid and it takes 2.5 hours for half of the fluid to escape.

(b) Determine the total time, correct to the nearest minute, that it takes for the leak to stop. (4 marks)

$Q(0) = Ae^0 + 100$

$900 = A + 100$

$\Rightarrow A = 800$ ✓ calculates value for A.

$Q(150) = 800e^{-150k} + 100$

$450 = 800e^{-150k} + 100$

$k = 0.0055$ (4dp) ✓ calculates 'k' to appropriate level of dp.

$\therefore Q(t) = 800e^{-0.0055t} + 100$

$t \to 0$ but get as close as possible. as $t \neq 0$.

writes equation to solve

$100 + 800e^{-0.0055t} < 100$.

or $800e^{-0.0055t} \approx 0$.

$\therefore t = 1345.08$

$t \approx 1345$ mins.

solution for 't' to the nearest minute

(accept other t values if relevant/correct working)

Question 9 **(10 marks)**

The lifetime of antibiotics in the bloodstream of a patient is believed to be distributed as a logarithmic random variable, with mean $\mu = 8$ hours and a standard deviation of $\sigma = 8$ hours.

A random sample of the bloodwork of 40 patients is selected to analyse the speed at which the human body metabolises a particular type of antibiotic that is being trialled. The variable $\bar{X}$ represents the mean lifetime of the antibiotic in the bloodstream for these 40 patients.

(a) State the distribution of the sample mean lifetime $\bar{X}$. Justify your answer. (3 marks)

$\bar{X}$ is approximately normally distributed as the sample size, $n = 40$, which is > 30. ✓ makes statement about sample size

$$\bar{X} \sim N\left(8, \frac{8^2}{40}\right)$$

$$\bar{X} \sim N(8, 1 \cdot 6)$$ ✓ makes normally distribution statement

$$\sigma(\bar{X}) = \sqrt{1 \cdot 6} = 1 \cdot 2649.$$ ✓ states st. dev.

(b) Determine the probability that the sample mean lifetime is between 5 and 11 hours. (1 mark)

$P(5 < \bar{X} < 11) = 0 \cdot 9823$ ✓ correct answer.

Question 9 continued

Kathryn, the chief doctor in charge of the study, suggests that the lifetimes may not be logarithmically distributed, and that an alternative distribution would still have a mean of $\mu = 8$ hours and a standard deviation of $\sigma = 8$ hours.

(c) If Kathryn is correct, will your answer to (b) change? Explain your answer. (2 marks)

There will be no change. ✓ States no change.

The sample size of 40 is > 30, which provides a normal distribution for the sample means, despite the shape of the original distribution ✓ valid reason with reference to sample size.

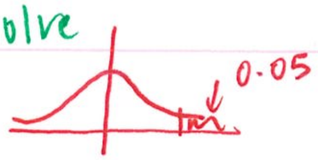
or since $n \geq 30$, we could still use a normal distribution to approximate the distribution of $\bar{X}$.

$\therefore$ There will be no change.

A different random sample of size n of the bloodwork of patients was selected. Repeated sampling of samples of size n shows that there is a 5% chance of obtaining a sample mean greater than 10 hours.

(d) Determine the value of n. (4 marks)

$P(\bar{X} > 10) = 0.05$ ✓ writes statement to solve



$\bar{X} \sim N\left(8, \frac{8^2}{n}\right)$ ie $\sigma(\bar{X}) = \frac{8}{\sqrt{n}}$

If $P(Z > k) = 0.05$ $\therefore$ $k = 1.6448$ ✓ calculates K value

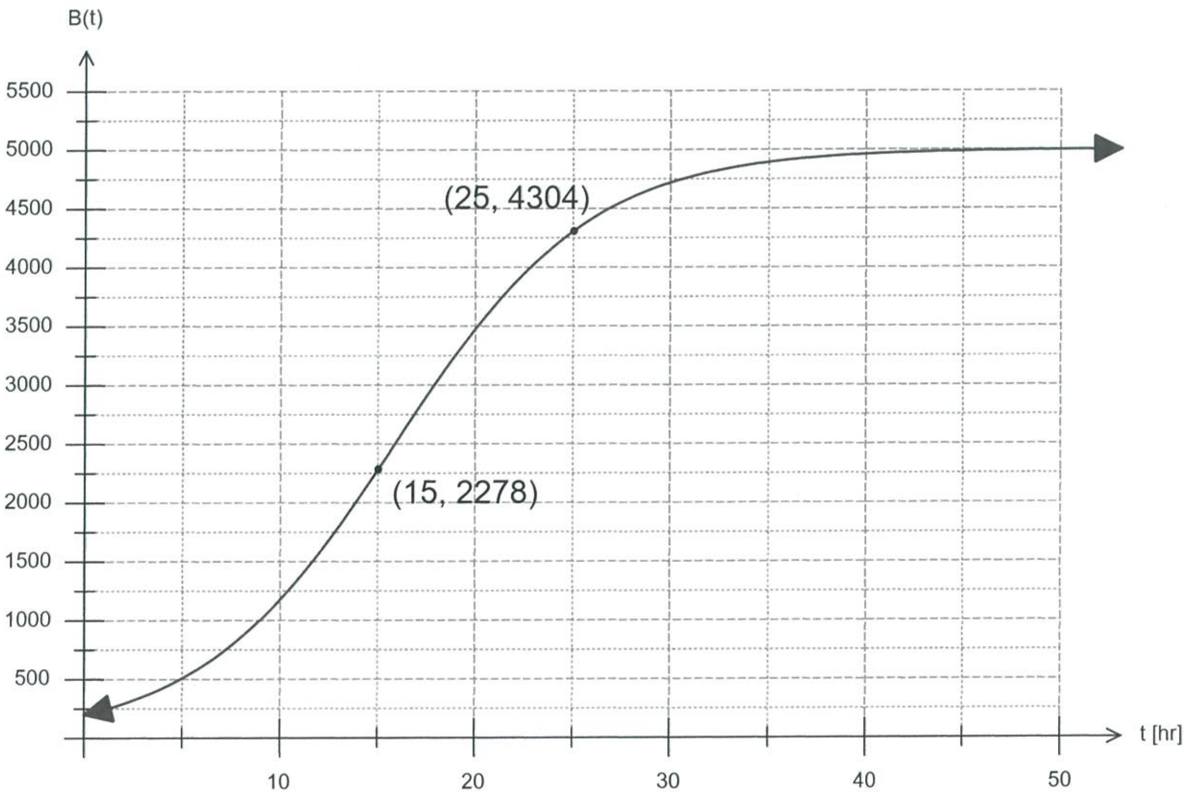
$$\frac{10-8}{\frac{8}{\sqrt{n}}} = 1.6448$$ ✓ writes equation to solve for 'n'

$n = 43.29$

So $n = 44$ ✓ obtains correct value for 'n' with correct rounding.

Question 10 **(7 marks)**

A scientific experiment measures the total bacteria present per hour over a few days. The graph below represents the amount of bacteria $B(t)$ present for t hours. Two particular measurements are provided for $t = 15$ hours and $t = 25$ hours.



(a) Suggest a differential equation that would model $B(t)$, the number of bacteria present per hour. (1 mark)

logistic equation

$$\therefore \frac{dB}{dt} = rB(K - B) \checkmark$$

writes valid equation.

where 'r' and 'K' are constants.

Question 10 continued

(b) Determine the solution $B(t)$ to the differential equation in (a). (4 marks)

5000 is the limit

$$B(t) = \frac{5000}{1 + Ae^{-kt}}$$

✓ writes initial equation

$$B(15) = 2278 = \frac{5000}{1 + Ae^{-15k}}$$

$$B(25) = 4304 = \frac{5000}{1 + Ae^{-25k}}$$

✓ obtains/writes equations ~~for~~ to solve for A & k.

$$\therefore A = 24, \quad k = 0.2$$ ✓

Solves for A and K.

$$\therefore B(t) = \frac{5000}{1 + 24e^{-0.2t}}$$

✓ correct final equation for B(t).

(c) How long would it take for the bacteria to reach its maximum capacity? (3 marks)

max = 5000, but doesn't get there exactly.

$$B(t) = \frac{5000}{1 + 24e^{-0.2t}} < 5000$$

✓ makes inequality to solve for t.

$$\Rightarrow \frac{5000}{1 + 24e^{-0.2t}} = 4999.99$$

✓ writes equation to solve to find 't'

or

$$24e^{-0.2t} > 0$$

$$t = 61.94$$ ✓

(accept 61 hrs 56 mins)

solves for 't'. accept 1, 2 dp for rounding

Answer only 1 ~~2~~ marks.

Note: 't' can be different values depending on accuracy of value to solve for. Accept other values if appropriate working.
eg $B(t) = 4999.9907$ $t = 81.87$ hrs.

End of questions